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Fokker–Planck equation associated with the Wigner function of a quantum system with a finite number of states

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Abstract. We consider a quantum system with a finite number N of states and we show that a Markov process evolving in an 'extended' discrete phase space can be associated with the discrete Wigner function of the system. This Wigner function is built using the Weyl quantisation procedure on the group $\mathbb{Z}_N \times \mathbb{Z}_N$. Moreover, we can use this process to compute the quantum mean values as probabilistic expectations of functions of this process. This probabilistic formulation can be seen as a stochastic mechanics in phase space.

1. Introduction

The interest in using Wigner functions in the probabilistic description of evolutions of quantum systems has been emphasised many times (see, e.g., [1, 2]). In particular, they can be used in the framework of stochastic mechanics [3, 4] where they allow one to treat pure or mixed states at the same level.

The present work is 'in the line' of previous results obtained, for example, in [5-7] where the density of a quantum state is shown to be solution of a forward Kolmogorov (Fokker-Planck) equation. This equation leads to a Markov process taking values in the configuration space in [6, 7], and in the momentum space in [5]. We want to give here a similar probabilistic interpretation of the Weyl-Wigner formulation of quantum mechanics. But it is well known that the Wigner function is not a probability density. To give it a precise probabilistic meaning we introduce a decomposition of the Wigner function as a difference of two positive functions. This couple defines the density of a stochastic process having values in the phase space enlarged by a dichotomic variable. This programme can be completed in the continuous case, at least formally [8]. Moreover, if the system has an associated Hilbert space of finite dimension, rigorous results can be formulated. In [3] a first approach to this problem has already been presented. In the following we give general results, in particular the problem concerning the case of a Hilbert space with an even dimension is completely solved by using group theory considerations. The theory of discrete Weyl and Fano operators we develop here, in order to define a discrete Wigner function, is mainly based on the properties of the phase space group $\mathbb{Z}_N \times \mathbb{Z}_N$.

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2. Weyl system on an Abelian group

In order to fix the notation we recall first some basic properties of Weyl systems. These systems can be defined in the following abstract way.

2.1. Projective representation

Let (G, +) be a (locally compact) Abelian group. A multiplier on G is a function m such that

$$m : \mathrm{G}\times\mathrm{G}\to\mathrm{T}\equiv\{z\in\mathbb{C};\ |z|=1\}$$

$$m(0,0)=1$$

$$m(X_1,X_2)m(X_1+X_2,X_3)=m(X_1,X_2+X_3)m(X_2,X_3) \qquad \forall X_1,X_2,X_3\in\mathrm{G}.$$

m satisfies

$$m(0,X)=m(X,0) \qquad m(X,-X)=m(-X,X) \qquad \forall X\in\mathrm{G}.$$

A unitary projective representation P of G with respect to a multiplier m on G is a mapping from G to the set of unitary operators acting on a Hilbert space, such that

$$P(X)P(X')=m(X,X')P(X+X') \qquad \forall X,X'\in\mathrm{G}.$$

The $P(X)$ have the following commutation relations:

$$P(X)P(X')=b_m(X,X')P(X')P(X)$$

where b_m is the antisymmetric bicharacter of G associated with m

$$b_m(X,X')\equiv m(X,X')\bar{m}(X',X) \tag{2.1}$$

(the bar denotes complex conjugation).

Conversely, with the antisymmetric bicharacter b_m defined in (2.1), we can associate a class of multipliers. More precisely, we have the following result.

Theorem 2.1 [9]. If to two multipliers m and n on the same group G there corresponds a common bicharacter then there exists a function η from G to T such that

$$m(X,X')=\eta(X)\eta(X')\bar{\eta}(X+X')n(X,X') \qquad \forall X,X'\in\mathrm{G}.$$

2.2. Phase space formulation

Let $\mathscr{G}$ be a (locally compact) Abelian group. The dual group $\hat{\mathscr{G}}$ of $\mathscr{G}$ (the Abelian group of all the non-equivalent irreducible unitary representations of $\mathscr{G}$), can be identified with the set of characters of $\mathscr{G}$. For any $x\in\hat{\mathscr{G}}$ there exists a character $\chi_x:\mathscr{G}\to\mathrm{T}$ such that

$$\chi_x(y)\chi_x(y')=\chi_x(y+y') \qquad \chi_x(y)\chi_{x'}(y)=\chi_{x+x'}(y) \qquad \forall y,y'\in\mathscr{G} \quad x'\in\hat{\mathscr{G}}.$$

Definition 2.2. We denote the product group $\hat{\mathscr{G}}\times\mathscr{G}$ by G. Let m be a multiplier on G. We shall use the expression 'a Weyl system on the group G with respect to the multiplier m' to refer to a unitary projective representation W of G on a Hilbert space H.

Using the concise notation $W(X) \equiv W_X$ for the Weyl operators, we have

$$\begin{aligned} W_X W_{X'} &= m(X, X') W_{X+X'} \\ W_0 &= \mathbb{1}_H \end{aligned} \tag{2.2}$$

and

$$W_X^{\dagger} = W_X^{-1} = \bar{m}(X, -X) W_{-X}$$

(here † denotes the adjoint).

In what follows we shall always suppose that the multiplier m has the particular associated bicharacter which is equivalent to that used in continuous quantum mechanics

$$b_m(X, X') = \chi_x(y')\bar{\chi}_{x'}(y) \tag{2.3}$$

with $X = (x, y)$ and $X' = (x', y') \in \mathrm{G}$.

We have to notice that if we define $n(X, X') \equiv \bar{\chi}_{x'}(y)$, then n is clearly a multiplier and $b_n = b_m$. Then there exists a function η from G to T such that

$$m(X, X') = \eta(X)\eta(X')\bar{\eta}(X + X')\bar{\chi}_{x'}(y). \tag{2.4}$$

Theorem 2.3. For a Weyl system defined as above there exists a function η from G to T and two families of unitary operators on H, V_x, $x \in \hat{\mathscr{G}}$ and U_y, $y \in \mathscr{G}$, representations of $\hat{\mathscr{G}}$ and $\mathscr{G}$ respectively, with the commutation relations

$$V_x U_y = \chi_x(y) U_y V_x \tag{2.5}$$

such that

$$W_{xy} = \eta(x, y) V_x U_y \tag{2.6}$$

for all $(x, y) \in \mathrm{G}$. The family $\{V_x, U_y\}$ is then what is known as a representation of the canonical commutation relations (CCR) in the Weyl form.

Proof. The function η has been previously defined by (2.4), then if we set for $(x, y) \in \mathrm{G}$

$$V_x = \bar{\eta}(x, 0) W_{x0} \qquad U_y = \bar{\eta}(0, y) W_{0y}$$

then the V and the U satisfy

$$V_x V_{x'} = V_{x+x'} \tag{2.7}$$

$$U_y U_{y'} = U_{y+y'} \tag{2.8}$$

and (2.5) and (2.6) are valid. □

Conversely if the V and the U satisfy (2.5), (2.7) and (2.8) then to the W defined by (2.6) correspond a Weyl system, with multiplier m given by (2.4), for any function η.

3. Quantum systems with a finite number of states

3.1. Discrete Weyl operators

Consider a quantum system with a finite number N of states. The Hilbert space associated with this system is $\mathbb{C}^N$ which is isomorphic to the space of periodic functions of period N from $\mathbb{Z}$ to $\mathbb{C}$. We shall denote this space by $C(\mathbb{Z}_N)$ where $\mathbb{Z}_N$ is the group of residues modulo N of $\mathbb{Z}$, isomorphic to the cyclic group of order N. The state space $C(\mathbb{Z}_N)$ is endowed with the usual scalar product

$$(\varphi, \psi) = \sum_{x \in \mathbb{Z}_N} \bar{\varphi}(x)\psi(x)$$

for φ and ψ belonging to $C(\mathbb{Z}_N)$. The dual group $\hat{\mathbb{Z}}_N$ of $\mathbb{Z}_N$ is isomorphic to $\mathbb{Z}_N$ and we identify them. In order to build an explicit Weyl system we choose

$$\chi_x(y) = \xi^{xy}$$

for x, y in $\mathbb{Z}_N$ and with $\xi = \exp(2i\pi/N)$ and we define the Fourier transform $\mathscr{F}$ on $C(\mathbb{Z}_N)$ by

$$(\mathscr{F}\psi)(x) = N^{-1/2} \sum_{y \in \mathbb{Z}_N} \xi^{-xy}\psi(y).$$

Now we introduce the unitary operators $\boldsymbol{V}$ and $\boldsymbol{U}$ from $C(\mathbb{Z}_N)$ to $C(\mathbb{Z}_N)$, tied to the phase multiplication and to the left translation

$$(\boldsymbol{V}\psi)(x) = \xi^x\psi(x) \qquad (\boldsymbol{U}\psi)(x) = \psi(x-1).$$

Taking $\boldsymbol{V}_x \equiv \boldsymbol{V}^x$, $x \in \mathbb{Z}_N$, the xth power of $\boldsymbol{V}$, and $\boldsymbol{U}_y \equiv \boldsymbol{U}^y$, $y \in \mathbb{Z}_N$, the yth power of $\boldsymbol{U}$, then $\{\boldsymbol{V}_x, \boldsymbol{U}_y\}$ satisfies the properties (2.7), (2.8) and (2.5)

$$\boldsymbol{V}_x\boldsymbol{U}_y = \xi^{xy}\boldsymbol{U}_y\boldsymbol{V}_x.$$

So they give a representation of the CCR over $\mathbb{Z}_N$. Moreover, by the Stone–Von Neumann–Mackey theorem [10] every irreducible representation of the CCR over $\mathbb{Z}_N$ is isometrically equivalent to this one.

Therefore the operators

$$\boldsymbol{W}_{xy} = \eta(x, y)\boldsymbol{V}^x\boldsymbol{U}^y \tag{3.1}$$

are Weyl operators for any fixed function $\eta : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathrm{T}$.

One can easily see from (2.2) and (2.3) that

$$\mathrm{Tr}(\boldsymbol{W}_X^\dagger \boldsymbol{W}_{X'}) = N\delta_X(X') \tag{3.2}$$

for every X and X' of $\mathbb{Z}_N^2$. $\delta_X(X') = 1$ if $X = X'$ and $\delta_X(X') = 0$ otherwise.

The family $\{\boldsymbol{W}_X, X \in \mathbb{Z}_N^2\}$ is thus a family of orthogonal operators with respect to the Hilbert–Schmidt scalar product. It gives a basis of the space $\mathscr{B}(C(\mathbb{Z}_N))$ of the linear operators on $C(\mathbb{Z}_N)$. Suppose $\boldsymbol{A}$ is a given operator of $\mathscr{B}(C(\mathbb{Z}_N))$, then there exists a complex function $\hat{A}(\ ,\ )$ on $\mathbb{Z}_N^2$ such that

$$\boldsymbol{A} = \frac{1}{N} \sum_{(x,y) \in \mathbb{Z}_N^2} \hat{A}(x, y)\boldsymbol{W}_{xy}. \tag{3.3}$$

3.2. Discrete Wigner functions

Following the ordinary Weyl quantisation procedure, we develop in the discrete case the method already used for the continuous case.

Let us define the discrete Fano operators

$$\Delta_{pq} = \frac{1}{N} \sum_{(x,y) \in \mathbb{Z}_N^2} \xi^{py-qx}\boldsymbol{W}_{xy} \tag{3.4}$$

with $(p, q) \in \mathbb{Z}_N^2$. Then with any classical observable $A(\ ,\ )$, a real function on $\mathbb{Z}_N^2$, we associate the 'quantum' operator of $\mathscr{B}(C(\mathbb{Z}_N))$

$$A = \frac{1}{N} \sum_{(p,q) \in \mathbb{Z}_N^2} A(p, q) \Delta_{pq}. \tag{3.5}$$

For any pure state $\psi \in C(\mathbb{Z}_N)$ of the system we can then write the mean value of A in the form

$$(\psi, A\psi) = \sum_{(p,q) \in \mathbb{Z}_N^2} A(p, q) f_\psi(p, q) \tag{3.6}$$

where $f_\psi(p, q) = (1/N)(\psi, \Delta_{pq}\psi)$ is the so-called discrete Wigner function.

We have seen that the family $\{W_{xy}, (x, y) \in \mathbb{Z}_N^2\}$ is a basis of $\mathscr{B}(C(\mathbb{Z}_N))$. But the Fano operators are such that

$$\Delta_{00} = \sum_{(x,y) \in \mathbb{Z}_N^2} W_{xy} \qquad \Delta_{pq} = W_{pq} \Delta_{00} W_{pq}^{\dagger} \qquad \operatorname{Tr}(\Delta_{pq}^{\dagger} \Delta_{p'q'}) = N \delta_p(p') \delta_q(q'). \tag{3.7}$$

The family $\{\Delta_{pq}, (p, q) \in \mathbb{Z}_N^2\}$ is also a basis of $\mathscr{B}(C(\mathbb{Z}_N))$ and any self-adjoint operator can be written in the form (3.3) or (3.5) with

$$A(p, q) = \operatorname{Tr}(\Delta_{pq}^{\dagger} A) = \frac{1}{N} \sum_{(x,y) \in \mathbb{Z}_N^2} \xi^{-py+qx} \hat{A}(x, y)$$

$$\hat{A}(x, y) = \operatorname{Tr}(W_{xy}^{\dagger} A) = \frac{1}{N} \sum_{(p,q) \in \mathbb{Z}_N^2} \xi^{py-qx} A(p, q)$$

A and $\hat{A}$ being 'symplectic' Fourier transforms of one another.

In order to have a real mean value of the operator A, f_ψ has to be real for any ψ. This condition will be fulfilled if and only if

$$\Delta_{pq} = \Delta_{pq}^{\dagger}$$

for every $(p, q) \in \mathbb{Z}_N^2$. This is possible if and only if the function η given in (3.1) satisfies

$$\eta(x, y) \eta(-x, -y) \xi^{xy} = 1$$

for each $(x, y) \in \mathbb{Z}_N^2$.

Moreover, the ordinary probabilistic interpretation of quantum mechanics imposes (for normalised ψ)

$$\sum_{q \in \mathbb{Z}_N} f_\psi(p, q) = |(\mathscr{F}\psi)(p)|^2 \tag{3.8}$$

$$\sum_{p \in \mathbb{Z}_N} f_\psi(p, q) = |\psi(q)|^2. \tag{3.9}$$

So that f_ψ would satisfy

$$\sum_{(p,q) \in \mathbb{Z}_N^2} f_\psi(p, q) = 1.$$

These conditions can be fulfilled for every ψ if and only if

$$\eta(x, 0) = \eta(0, x) = 1$$

$\forall x \in \mathbb{Z}_N$. This last condition also implies that

$$\frac{1}{N} \sum_{(p,q) \in \mathbb{Z}_N^2} \Delta_{pq} = \mathbb{1}.$$

Therefore we have the following result.

Proposition 3.1. Let W be the Weyl system on $\mathbb{Z}_N^2$ given by (3.1), then the unique mapping $f: \psi \to f_\psi$ from $C(\mathbb{Z}_N)$ *to the space of functions on* $\mathbb{Z}_N^2$ such that (3.6) is satisfied for every ψ of $C(\mathbb{Z}_N)$ and every self-adjoint operator A, is given by

$$f_\psi(p, q) = \frac{1}{N}(\psi, \Delta_{pq}\psi)$$

where the Fano operators Δ_{pq} are defined by (3.4). Moreover, f_ψ will be real and will give the marginal distributions (3.8) and (3.9) for each ψ if and only if

$$\eta(x, y)\eta(-x, -y)\xi^{xy} = 1 \tag{3.10}$$

$$\eta(x, 0) = \eta(0, x) = 1 \tag{3.11}$$

for every $(x, y) \in \mathbb{Z}_N^2$.

Notes. The conditions (3.10) and (3.11) characterise the class of functions η which have to be used. For odd N we have already given an example of a convenient function η [3]. For any choice of N one can choose, for example,

$$\eta(x, y) = \xi^{-xy/2+[xy/N]N/2}$$

where $[r]$ designates the integer part of the real number r.

All the previous results can be extended without any difficulty to the case of mixed states. For that it is sufficient to define the Wigner function associated with a mixed state given by a density operator D (a positive operator of $\mathscr{B}(C(\mathbb{Z}_N))$ with trace 1) in the following way:

$$f_D(p, q) = \frac{1}{N}\operatorname{Tr}(\Delta_{pq}D).$$

Note that f_D completely determines the state of the system because from the Wigner function f_D we can reconstruct the operator D

$$D = \sum_{(p,q)\in\mathbb{Z}_N^2} f_D(p, q)\Delta_{pq}.$$

In all cases from (3.7) we deduce a bound of f_D which will be used in the following

$$|f_D(p, q)| \leq N^{-1/2}. \tag{3.12}$$

4. Markov process in the discrete phase space

The construction of the Markov process associated with the Wigner function we introduce here is a generalisation of the one we gave in [3]. The number of possible states N of the system is odd or even and the function η appearing in (3.1) satisfies the conditions (3.10) and (3.11) (even if (3.11) is not really necessary for what follows).

Let H be a self-adjoint operator from $C(\mathbb{Z}_N)$ to $C(\mathbb{Z}_N)$ and D_t, the density operator of the system at time t, be solution of

$$\partial_t D_t = \mathrm{i}[D_t, H] \tag{4.1}$$

with some initial condition $D_{t=0} = D_0$.

The Wigner function associated with $\boldsymbol{D}_t$ being by definition

$$f_t(X)=\frac{1}{N}\operatorname{Tr}(\Delta_X \boldsymbol{D}_t)$$

$X\in\mathbb{Z}_N^2$, we deduce from (4.1) that f_t satisfies the evolution equation

$$\partial_t f_t(X)=\sum_{X'\in\mathbb{Z}_N^2}\mathscr{H}(X,X')f_t(X') \tag{4.2}$$

where

$$\mathscr{H}(X,X')=\frac{\mathrm{i}}{N}\operatorname{Tr}([\Delta_X,\Delta_{X'}]\boldsymbol{H}). \tag{4.3}$$

It is easy to see that $\mathscr{H}$ satisfies

$$\sum_{X\in\mathbb{Z}_N^2}\mathscr{H}(X,X')=\sum_{X'\in\mathbb{Z}_N^2}\mathscr{H}(X,X')=\mathscr{H}(X,X)=0. \tag{4.4}$$

Starting from the Wigner function $f_t(X)$, we construct a strictly positive function $g_t(X,\sigma)$ defined on $\mathbb{Z}_N^2\times\{-1,1\}$ if we set

$$g_t(X,\sigma)=\frac{1}{2N^2}+\frac{1}{4N^{3/2}}\sigma f_t(X). \tag{4.5}$$

This function g_t has the following properties:

$$\sum_{(X,\sigma)\in\mathbb{Z}_N^2\times\{-1,1\}} g_t(X,\sigma)=1 \qquad 2N^{3/2}\sum_{\sigma\in\{-1,1\}}\sigma g_t(X,\sigma)=f_t(X). \tag{4.6}$$

Using the bound we gave in (3.12), we see that g_t is indeed strictly positive. Moreover, using (4.4) and (4.2), we have

$$\partial_t g_t(X,\sigma)=\sum_{(X',\sigma')\in\mathbb{Z}_N^2\times\{-1,1\}}\mathscr{H}(X,X')\delta_{\sigma'}(\sigma)g_t(X',\sigma'). \tag{4.7}$$

This equation can in fact be put in the form of a forward Kolmogorov equation (Fokker–Planck equation). Thus g_t can be interpreted as the probability distribution of a Markov process in $\mathbb{Z}_N^2\times\{-1,1\}$. Indeed if we introduce

$$A_t(X,\sigma;X',\sigma')=\frac{\mathscr{H}_{\mathrm{m}}}{g_t(X',\sigma')}+\mathscr{H}(X,X')\delta_{\sigma'}(\sigma) \qquad \text{if } (X,\sigma)\neq(X',\sigma') \tag{4.8}$$

$$A_t(X,\sigma;X,\sigma)=-(2N^2-1)\frac{\mathscr{H}_{\mathrm{m}}}{g_t(X,\sigma)} \tag{4.9}$$

where $\mathscr{H}_{\mathrm{m}}=\max_{(X,X')\in\mathbb{Z}_N^4}|\mathscr{H}(X,X')|$, we have from (4.7) that

$$\partial_t g_t(X,\sigma)=\sum_{(X',\sigma')\in\mathbb{Z}_N^2\times\{-1,1\}}A_t(X,\sigma;X',\sigma')g_t(X',\sigma'). \tag{4.10}$$

A_t is a Markov generator because if $(X,\sigma)\neq(X',\sigma')$ then

$$A_t(X,\sigma;X',\sigma')\geqslant 0 \qquad \sum_{(X,\sigma)\in\mathbb{Z}_N^2\times\{-1,1\}}A_t(X,\sigma;X',\sigma')=0.$$

Therefore the general theory of stochastic processes [11] gives the following theorem.

Theorem 4.1. One can find a probability space $(\Omega,\mathscr{F},\mu)$ such that the function A_t defined on $(\mathbb{Z}_N^2\times\{-1,1\})^2$, $t\in[0,T]$ an interval of $\mathbb{R}^+$, and given by (4.8) and (4.9) generates a Markov process $Z(t)=(X(t),\sigma(t))$ on $(\Omega,\mathscr{F},\mu)$, having values in $\mathbb{Z}_N^2\times\{-1,1\}$ and with transition probability

$$P(X,\sigma,t;X',\sigma',t')=\mu(Z(t)=(X,\sigma)/Z(t')=(X',\sigma')) \qquad t>t'$$

solution of the forward Kolmogorov equation

$$\partial_t P(X,\sigma,t;X',\sigma',t')=\sum_{(Y,\tau)\in\mathbb{Z}_N^2\times\{-1,1\}} A_t(X,\sigma;Y,\tau)P(Y,\tau,t;X',\sigma',t') \tag{4.11}$$

with the initial condition

$$\lim_{t\downarrow t'} P(X,\sigma,t;X',\sigma',t')=\delta_{X'}(X)\delta_{\sigma'}(\sigma). \tag{4.12}$$

In particular, if the initial distribution of $Z(t)$ is the function g_0, given by (4.5) with $t=0$, the distribution at time t of $Z(t)$ is the solution of (4.10), namely

$$g_t(X,\sigma)=\sum_{(X',\sigma')\in\mathbb{Z}_N^2\times\{-1,1\}} P(X,\sigma,t;X',\sigma',0)g_0(X',\sigma').$$

The stochastic process $Z(t)$ can be used to write the quantum mean value of any operator as a probabilistic mean value. Let $\boldsymbol{K}$ be any self-adjoint operator on $C(\mathbb{Z}_N)$, by definition the quantum mean value of $\boldsymbol{K}$ is

$$\langle\boldsymbol{K}\rangle=\mathrm{Tr}(\boldsymbol{K}\boldsymbol{D}_t)=\sum_{X\in\mathbb{Z}_N^2}\mathrm{Tr}(\Delta_X\boldsymbol{K})f_t(X).$$

Set

$$K(X,\sigma)=2N^{3/2}\sigma\,\mathrm{Tr}(\Delta_X\boldsymbol{K})$$

using (4.6) we deduce that the mean value of $\boldsymbol{K}$ is

$$\langle\boldsymbol{K}\rangle=\mathbb{E}K(Z(t)) \tag{4.13}$$

where $Z(t)$ is the Markov process defined by (4.11) and (4.12) and with initial distribution g_0.

Now one could ask how to recover a Wigner function from a Markov process having values in $\mathbb{Z}_N^2\times\{-1,1\}$. We give here a natural answer.

Let A_t be the Markov generator of some Markov process $Z(t)$ defined on $\mathbb{Z}_N^2\times\{-1,1\}$ and g_t the probability distribution corresponding to some initial distribution g_0. Define

$$f_t(X)=\sum_{\sigma\in\{-1,1\}}2N^{3/2}\sigma g_t(X,\sigma). \tag{4.14}$$

Suppose that at time $t=0$ f_t is a Wigner function (namely there exists a density operator $\boldsymbol{D}$ such that $f_0(X)=1/N\,\mathrm{Tr}(\Delta_x\boldsymbol{D})$); then f_t is a Wigner function at all times if we can find an operator $\boldsymbol{H}$ such that

$$\sum_{X',\sigma',\sigma}\sigma A_t(X,\sigma;X',\sigma')g_t(X',\sigma')=\sum_{X',\sigma'}\mathscr{H}(X,X')\sigma' g_t(X',\sigma')$$

where

$$\mathscr{H}(X,X')=\frac{\mathrm{i}}{N}\mathrm{Tr}([\Delta_X,\Delta_{X'}]\boldsymbol{H})$$

does not depend on t and σ'.

This is obviously a consequence of the fact that the function $f_t(X)$ in (4.14) is solution of an equation of type (4.2) and that one can prove, using Dyson's series, that f_t is indeed a Wigner function.

5. Another phase space formulation

We want to stress here some points of comparison between the phase space formulation we just gave and the one introduced by Varilly and Gracia-Bondia [12]. If the system has spin j then Varilly and Gracia-Bondia take the phase space equal to S^2 and show that it is possible to construct on S^2 a family $\{\Delta^j(n), n \in S^2\}$ of self-adjoint operators from $\mathbb{C}^{2j+1}$ to $\mathbb{C}^{2j+1}$ such that

$$\frac{2j+1}{4\pi}\int_{S^2} \Delta^j(n)\,\mathrm{d}n = I$$

$$\frac{2j+1}{4\pi}\int_{S^2} \mathrm{Tr}(\Delta^j(m)\Delta^j(n))\Delta^j(n)\,\mathrm{d}n = \Delta^j(m) \qquad \forall m \in S^2 \tag{5.1}$$

$$\Delta^j(g\cdot n) = \Pi_j(g)\Delta^j(n)\Pi_j(g)^{-1} \qquad \forall g \in \mathrm{SU}(2)$$

where $\mathrm{d}n = \mathrm{d}\varphi\,\mathrm{d}\cos\theta$ if (θ, φ) are the spherical coordinates of $n \in S^2$, $g\cdot n$ denotes the natural action of g on n and Π_j is an irreducible representation of SU(2) on $\mathbb{C}^{2j+1}$. The Stratonovich–Weyl symbol of an operator $\boldsymbol{A}$ from $\mathbb{C}^{2j+1}$ to $\mathbb{C}^{2j+1}$ is defined by

$$\mathscr{W}_A(n) = \mathrm{Tr}(\Delta^j(n)\boldsymbol{A})$$

and one has

$$\boldsymbol{A} = \frac{2j+1}{4\pi}\int_{S^2} \mathscr{W}_A(n)\Delta^j(n)\,\mathrm{d}n.$$

So if the system is in state $\boldsymbol{D}$, a density operator on $\mathbb{C}^{2j+1}$, the Wigner function on the sphere S^2 is by definition

$$f_{\boldsymbol{D}}(n) = \frac{2j+1}{4\pi}\mathrm{Tr}(\Delta^j(n)\boldsymbol{D}).$$

In fact one can go from this formulation to the one we have proposed or vice versa in a very simple way. For a spin $j = \frac{1}{2}(N-1)$, $N \in \mathbb{N}\backslash\{0\}$, the Fano operators will be defined from $\mathbb{C}^N = C(\mathbb{Z}_N)$ to $\mathbb{C}^N$. Thus for the operator $\boldsymbol{A}$ we will have

$$\boldsymbol{A} = \sum_{(x,y)\in\mathbb{Z}_N^2} A(x,y)\Delta_{xy} \tag{5.2}$$

with

$$A(x,y) = \frac{2j+1}{4\pi}\int_{S^2} \mathscr{W}_A(n)\Phi_{xy}(n)\,\mathrm{d}n$$

and where

$$\Phi_{xy}(n) = \mathrm{Tr}(\Delta^j(n)\Delta_{xy}).$$

Of course for any $\boldsymbol{A}$ given by (5.2) one has the converse formula

$$\mathscr{W}_A(n) = \frac{1}{N}\sum_{(x,y)\in\mathbb{Z}_N^2} A(x,y)\Phi_{xy}(n).$$

On the other hand one can also build on $S^2\times\{-1, 1\}$ a Markov process associated with the Wigner function on the sphere by using the same kind of technique we have used in section 4. As previously, this process allows one to compute quantum mean values in a probabilistic way.

But the major difference between these two schemes is, as noticed by Varilly [13], the invariance property. The Fano operators do not have the SU(2) invariance property (5.1) but instead are invariant relative to the discrete Heisenberg group and therefore they give a discrete version of the usual phase space formulation of quantum mechanics.

6. General remarks

The first remark to make is that the transition probabilities of the process we have introduced are state dependent. This comes from the fact that the Markov generator A_t defined by (4.8) and (4.9) depends on the density g_t. This seems to be a general feature of all Markov processes associated with a quantum dynamics via a forward Kolmogorov equation [5-7], and then is an apparent feature of stochastic mechanics.

On the other hand, from the Wigner function we could define some other probability distribution g_t. For example

$$g_t = \frac{1}{2N^2} + \sigma r f_t$$

where r is any real number strictly less than $(1/2N^2)$. We can even enlarge the phase space by allowing a variable σ to take more than two values. Moreover, for a particular choice of the function η one can imagine different generators A_t. Indeed when $N = 2a - 1$, $a \in \mathbb{N}\backslash\{0\}$, we have used in [3] the function

$$\eta(x, y) = \exp[-(2\mathrm{i}\pi/N)axy]$$

(ax mod N is the division by 2 of x in $\mathbb{Z}_N$), and shown that in that case one can build a Markov generator A_t which has a different form from the one we give in (4.8) and (4.9). These remarks prove that the process associated with a Wigner function is definitely not unique. The same remark has been made by Jaekel and Pignon about the Nelson construction [14].

In a completely independent way Galetti and De Toledo Piza [15] study the limit of N going to infinity for the discrete Wigner function with the following special choice of the function η:

$$\eta(x, y) = \exp[-(\mathrm{i}\pi/N)xy]$$

if $x, y \in \{-\frac{1}{2}(N-1), -\frac{1}{2}(N-1)+1, -\frac{1}{2}(N-1)+2, \ldots, \frac{1}{2}(N-1)\}$ and N is odd. Formally the discrete Wigner function goes to the one given by Wigner in [16] for the case where the Hilbert space is $L^2(\mathbb{R})$. The limit is achieved by the use of Riemann sums on a finite lattice with mesh length $(2\pi/N)^{1/2}$. Therefore the discrete Wigner function can be seen as a discretisation on a finite lattice of the usual Wigner function defined on $\mathbb{R}^2$.

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